Extracting universal properties of critical systems from reduced density matrix data

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Motivation

- DMRG allows for very accurate estimates of ground state energies in 1d many-body systems
- works by considering a finite region of length ℓ and truncating spectrum of the reduced density matrix (rather than spectrum of the hamiltonian), for $\ell \uparrow$
- less effective for correlation functions in the ground state since truncation breaks translational invariance, large effects near the boundaries, especially if $\xi \sim \ell$

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- less effective for correlation functions in the ground state since truncation breaks translational invariance, large effects near the boundaries, especially if $\xi \sim \ell$
- for the purpose of confronting an analytic theory with numerical results, it might be better to derive analytic results for correlations in the truncated space, or even better, in particular eigenstates of the density matrix

In this talk I will present some of these analytic results, for

- critical theories (CFT) in 1+1 dimensions
- gapped theories in 1+1 dimensions
- some limited results in higher dimensions
- en route I'll summarize the path integral approach to computing entanglement entropy

Basic setup

- extended quantum system, Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- $A =$ degrees of freedom in some subregion, $B =$ the rest
- whole system is in the ground state $|0\rangle$ of some local hamiltonian H
- reduced density matrix $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} |0\rangle\langle 0|$
- ρ_A has eigenvalues $\{\lambda\}$ and eigenstates $|\lambda\rangle \in \mathcal{H}_A$ with $\lambda \geq 0$ and $\sum \lambda = 1$

- Rényi entropies

$$S^{(n)} = (1-n)^{-1} \log \mathrm{Tr}\, \rho_A^n = (1-n)^{-1} \log \sum \lambda^n$$

- entanglement entropy

$$S = -\mathrm{Tr}\, \rho_A \log \rho_A = -\sum \lambda \log \lambda = \lim_{n \to 1} S^{(n)}$$

$$\mathrm{Tr}\,\rho_A^n = \sum_\lambda \lambda^n = \int_0^{\lambda_{\max}} p(\lambda)\lambda^n d\lambda$$

where $p(\lambda)$ is the density of eigenstates of ρ_A.

- as $n \to \infty$, $\mathrm{Tr}\,\rho_A^n \sim \lambda_{\max}^n$
- this has the form of a Mellin transform and so

$$p(\lambda) = \int \frac{\mathrm{Tr}\,\rho_A^n}{\lambda^{n+1}} \frac{dn}{2\pi i}$$

- if $\mathcal{O} = \prod_i \Phi_i(x_i)$ is some product of local observables with $x_i \in A$, can we say something about

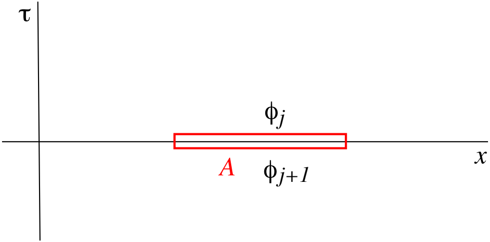
$$\langle \lambda | \mathcal{O} | \lambda \rangle \quad ?$$

$$\mathrm{Tr}\left(\mathcal{O}\rho_A^n\right) = \int_0^{\lambda_{\max}} \overline{\langle \lambda | \mathcal{O} | \lambda \rangle} p(\lambda) \lambda^n d\lambda$$

where $\overline{\langle \ldots \rangle}$ is the average over states in $(\lambda, \lambda + d\lambda)$

- if we can compute the lhs we can get information about these averages and in particular, taking $n \to \infty$, expectation values in $|\lambda_{\max}\rangle$

Path integral formulation



$$\mathrm{Tr}\,\rho_A^n = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_n (\langle\phi_1|\rho_A|\phi_2\rangle \ldots \langle\phi_n|\rho_A|\phi_1\rangle) = \frac{Z_{\mathcal{C}_n}}{Z^n}$$

where $Z_{\mathcal{C}_n}$ is the partition function on an n-sheeted cover of space-imaginary time, with conical singularities on $\partial A \cap \{\tau = 0\}$

Density of states

- Calabrese and Lefevre [2008] observed that in several simple cases for critical and near-critical systems in 1+1 dimensions
$$\mathrm{Tr}\,\rho_A^n \sim g^{n-1} e^{-b(n-1/n)}$$
where $b \propto c\log(L/a)$, $L \sim$ length of A or correlation length ξ whichever is smaller
- $-\log\lambda_{\mathrm{max}} \sim b - \log g$
- as long as λ is not too close to λ_{max}, saddle-point approximation gives
$$\lambda p(\lambda) \sim \frac{1}{2\sqrt{\pi}}\left(\frac{b}{\log(\lambda_{\mathrm{max}}/\lambda)}\right)^{1/4} e^{\sqrt{b\log(\lambda_{\mathrm{max}}/\lambda)}}$$
- in the same spirit we will apply their method to $\mathrm{Tr}(\mathcal{O}\rho_A^n)$.

Example 1. $A =$ finite interval in 1+1-dimensional CFT

- suppose A is the interval (x_1, x_2) and Φ is a local observable such that, in the infinite system

$$\langle 0|\Phi(\zeta)\Phi(\zeta')|0\rangle \sim |\zeta - \zeta'|^{-2\Delta}$$

- the mapping

$$z \to \zeta = \left(\frac{z - x_1}{x_2 - z}\right)^{1/n}$$

sends $\mathcal{C}_n \to \mathbb{C}$ and

$$\langle\Phi(x)\Phi(x')\rangle_{\mathcal{C}_n} = \left|\frac{d\zeta}{dx}\right|^{\Delta} \left|\frac{d\zeta'}{dx'}\right|^{\Delta} |\zeta - \zeta'|^{-2\Delta}$$

- the final expression is rather complicated but it simplifies as $n \to \infty$:

$$\langle \lambda_{\max} | \Phi(x)\Phi(x') | \lambda_{\max} \rangle \sim \left[\frac{(x_2 - x_1)^2}{(x - x_1)(x_2 - x)(x' - x_1)(x_2 - x') \log\left(\frac{(x - x_1)(x_2 - x')}{(x_2 - x)(x' - x_1)} \right)} \right]^{\Delta}$$

- this behaves like $|x_1 - x_2|^{-2\Delta}$ in the middle of the interval but also shows the singular behaviour near the ends
- Δ can be extracted knowing only $|\lambda_{\max}\rangle$

We find, for $\lambda < \lambda_{\max}$

$$\overline{\langle\lambda|\Phi(x)\Phi(x')|\lambda\rangle} \sim e^{\sqrt{B\log(\lambda_{\max}/\lambda)}-\sqrt{b\log(\lambda_{\max}/\lambda)}}$$

where $B = b + \Delta\log[(x - x_1)(x_2 - x')/(x_2 - x_1)^2]$, $b = -\log\lambda_{\max}$

- note that close to the end points $B < 0$ and the behaviour becomes oscillatory as a function of x and x'

Excess of the hamiltonian density in an interval

- suppose $H = \sum_j h(x_j) \sim \int h(x)dx$ where $h(x) = T_{tt}(x)$
- on the full line, this is normalised so that $\langle h(x) \rangle = 0$, but this is no longer the case in eigenstates of ρ_A

$$\frac{\int p(\lambda)\overline{\langle\lambda|h(x)|\lambda\rangle}\lambda^n d\lambda}{\int p(\lambda)\lambda^n d\lambda} = \langle T_{tt}(x)\rangle_{\mathcal{C}_n} = \frac{c(1-1/n^2)}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}$$

So $\quad \langle\lambda_{\mathrm{max}}|h(x)|\lambda_{\mathrm{max}}\rangle = \frac{c}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}$

- universal behaviour for $\lambda \leq \lambda_{\mathrm{max}}$

$$\overline{\langle\lambda|h(x)|\lambda\rangle} = \frac{c}{12\pi}\frac{(x_1-x_2)^2}{(x-x_1)^2(x_2-x)^2}\left(2-\frac{\log\lambda}{\log\lambda_{\mathrm{max}}}\right)$$

- changes sign at $\lambda \approx \lambda_{\mathrm{max}}^2$

For a semi-infinite interval $A = (0, \infty)$ in a gapped theory,

$$\int_0^\infty \overline{\langle \lambda | h(x) | \lambda \rangle} x dx \sim \frac{c \log \xi}{12\pi} \left(2 - \frac{\log \lambda}{\log \lambda_{\mathrm{max}}} \right)$$

[NB this assumes $v = 1$]

In higher dimensions, if A is the interior of a sphere of radius R, the excess hamiltonian density inside the sphere

$$\langle h(r,0)\rangle_{\mathcal{C}_n} = a_n \left(\frac{2R}{R^2 - r^2}\right)^d$$

where the a_n are universal, with a_∞ finite and $a_1 = 0$. If we assume $a_n \propto (1 - n^{-d})$ we get

$$\overline{\langle\lambda|h(x)|\lambda\rangle} = a_\infty \left(\frac{2R}{R^2 - r^2}\right)^d \left(d - (d-1)\frac{\log\lambda}{\log\lambda_{\max}}\right)$$

Summary

- in 1+1dimensions, critical exponents and the central charge c may in principle be extracted even if the reduced density matrix is truncated to a few (just one!) eigenstates

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- in 1+1dimensions, critical exponents and the central charge c may in principle be extracted even if the reduced density matrix is truncated to a few (just one!) eigenstates

- but this only half the talk! Does it work in practice?
- 'unusual' corrections to scaling $L_{\rm eff}^{-\Delta/n}$ may get in the way